Positive definiteness of fourth order three dimensional symmetric tensors

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Abstract

For a 4th order 3-dimensional symmetric tensor with its entries 1 or -1, we show the analytic sufficient and necessary conditions of its positive definiteness. By applying these conclusions, several strict inequalities is bulit for ternary quartic homogeneous polynomials.

Keywords: Positive definiteness, Fourth order tensors, Homogeneous polynomial.

1. Introduction

One of the most direct applications of positive definite tensors is to verify the vacuum stability of the Higgs scalar potential model [1, 2]. Qi [3] first used the consept of positive definiteness for a symmetric tensor when the order is even integer.

Definition 1.1. *Let $\mathcal{T} = (t_{i_1i_2\cdots i_m})$ be an mth order n dimensional symmetric tensor. $\mathcal{T}$ is called*

(i) **positive semi-definite** *([3]) if m is an even number and in the Euclidean space $\mathbb{R}^n$, its associated Homogeneous polynomial*

$$\mathcal{T}x^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} t_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m} \geq 0;$$

(ii) **positive definite** *([3]) if m is an even number and $\mathcal{T}x^m > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.*

Clearly, a positive semi-definite tensor coincides with a positive semi-definite matrix if $m = 2$. It is well-known that Sylvester's Criterion can efficiently identify the positive (semi-)definiteness of a matrix. The positive definiteness of a 4th order 2 dimensional symmetric tensor, (or positivity condition of a quartic univariate polynomial) may trace back to ones of Refs. Rees [4] , Lazard [5] Gadem-Li [6], Ku [7] and Jury-Mansour [8]. Untill to 2005, Wang-Qi [9] improved their proof, and perfectly gave analytic necessary and sufficient conditions. However, the above

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result depends on the discriminant of such a quartic polynomial. Hasan-Hasa [10] claimed that a necessary and sufficient condition of positive definiteness was proved without the discriminant. However, there is a problem in their argumentations. In 1998, Fu [11] pointed out that Hasan-Hasan's results are sufficient only. Recently, Guo[12] showed a new necessary and sufficient condition without the discriminant. Very recently, Qi-Song-Zhang[13] gave a new necessary and sufficient condition other than the above results. For more detail about applications of these results, see Song-Qi [14] also.

In 2005, Qi [3] gave that the sign of all H-(Z-)eigenvalue of a even order symmetric tensor can verify the positive definitiveness of such a higher order tensor. Subsequently, Ni-Qi-Wang [15] provided a method of computing the smallest eigenvalue for checking positive definiteness of a 4th order 3 dimensional tensor. Ng-Qi-Zhou [16] presented an algorithm of the largest eigenvalue of a nonnegative tensor. For a 4th order 3 dimensional symmetric tensor, Song [17] proved several sufficient conditions of its positive definiteness. Until now, an analytic necessary and sufficient condition has not been found for positive (semi-)definiteness for a 4th order 3 dimensional symmetric tensor.

In this paper, we mainly dicuss analytic necessary and sufficient conditions of positive definiteness of a class of 4th order 3-dimensional symmetric tensors (Theorem 3.1). Furthermore, several strict inequalities of ternary quartic homogeneous polynomial (Corollary 3.2) are built.

2. Copositivity of 4th order 2-dimensional symmetric tensors

Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor. Then for $x = (x_1, x_2)^\top$,

$$Tx^4 = t_{1111}x_1^4 + 4t_{1112}x_1^3x_2 + 6t_{1122}x_1^2x_2^2 + 4t_{1222}x_1x_2^3 + t_{2222}x_2^4. \tag{2.1}$$

Let

$$\begin{aligned}\Delta =&4 \times 12^3(t_{1111}t_{2222} - 4t_{1112}t_{1222} + 3t_{1122}^2)^3\\ &- 72^2 \times 6^2(t_{1111}t_{1122}t_{2222} + 2t_{1112}t_{1122}t_{1222} - t_{1122}^3 - t_{1111}t_{1222}^2 - t_{1112}^2t_{2222})^2\\ =&4 \times 12^3(I^3 - 27J^2),\end{aligned}$$

where

$$\begin{aligned}I =&t_{1111}t_{2222} - 4t_{1112}t_{1222} + 3t_{1122}^2,\\ J =&t_{1111}t_{1122}t_{2222} + 2t_{1112}t_{1122}t_{1222} - t_{1122}^3 - t_{1111}t_{1222}^2 - t_{1112}^2t_{2222}.\end{aligned}$$

and hence, the sign of Δ is the same as one of $(I^3 - 27J^2)$. Ulrich-Watson [18] presented the analytic conditions of the nonnegativity of a quartic and univariate polynomial in $\mathbb{R}_+$. Qi-Song-Zhang [13] also gave the nonnegativity and positivity of a quartic and univariate polynomial in $\mathbb{R}$, which means the positive (semi-)definitiveness of 4th order 2-dimensional tensor [2].

Lemma 2.1 ([2, 13]). *A 4th-order 2-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ is positive definite*

if and only if

$$(\boldsymbol{I}) \qquad \begin{cases} I^3 - 27J^2 = 0,\ \ t_{1112}\sqrt{t_{2222}} = t_{1222}\sqrt{t_{1111}}, \\ 2t_{1112}^2 + t_{1111}\sqrt{t_{1111}t_{2222}} = 3t_{1111}t_{1122} < 3t_{1111}\sqrt{t_{1111}t_{2222}}; \\ I^3 - 27J^2 > 0,\ \ |t_{1112}\sqrt{t_{2222}} - t_{1222}\sqrt{t_{1111}}| \le \sqrt{6t_{1111}t_{1122}t_{2222} + 2\sqrt{(t_{1111}t_{2222})^3}}, \\ (i)\ -\sqrt{t_{1111}t_{2222}} < 3t_{1221} \le 3\sqrt{t_{1111}v_{2222}}; \\ (ii)\ t_{1221} > \sqrt{t_{1111}t_{2222}}\ \ \textit{and} \\ |t_{1112}\sqrt{t_{2222}} + t_{1222}\sqrt{t_{1111}}| \le \sqrt{6t_{1111}t_{1122}t_{2222} - 2\sqrt{(t_{1111}t_{2222})^3}}. \end{cases}$$

A 4th-order 2-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ is positive semidefinite if and only if

$$(\boldsymbol{II}) \qquad \begin{cases} I^3 - 27J^2 \ge 0,\ \ |t_{1112}\sqrt{t_{2222}} - t_{1222}\sqrt{t_{1111}}| \le \sqrt{6t_{1111}t_{1122}t_{2222} + 2\sqrt{(t_{1111}t_{2222})^3}}, \\ (i)\ -\sqrt{t_{1111}t_{2222}} \le 3t_{1122} \le 3\sqrt{t_{1111}t_{2222}}; \\ (ii)\ t_{1122} > \sqrt{t_{1111}t_{2222}}\ \textit{and} \\ |t_{1112}\sqrt{t_{2222}} + t_{1222}\sqrt{t_{1111}}| \le \sqrt{6t_{1111}t_{1122}t_{2222} - 2\sqrt{(t_{1111}t_{2222})^3}}. \end{cases}$$

Lemma 2.2. *Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with its entires $|t_{ijkl}| = 1$ and $t_{1111} = t_{2222} = 1$. Then*

(i) $\mathcal{T}$ is positive semidefinite if and only if $t_{1122} = 1$;

(ii) $\mathcal{T}$ is positive definite if and only if $t_{1122} = 1$ and $t_{1112}t_{1222} = -1$.

Proof. (i) It follows from Lemma 2.1 (II) that $\mathcal{T}$ is positive semidefinite if and only if

$$I^3 - 27J^2 \ge 0,\ \ |t_{1112} - t_{1222}| \le \sqrt{6t_{1122} + 2} \text{ and } -1 \le 3t_{1122} \le 3.$$

Since $|t_{ijkl}| = 1$, then which means $t_{1122} = 1$ and either $t_{1112}t_{1222} = 1$,

$$I^3 - 27J^2 = (1 - 4 + 3)^3 - 27(1 + 2 - 1 - 1 - 1)^2 = 0,$$

$$|t_{1112} - t_{1222}| = 0 < \sqrt{6t_{1122} + 2} = \sqrt{8};$$

or $t_{1112}t_{1222} = -1$,

$$I^3 - 27J^2 = (1 + 4 + 3)^3 - 27(1 - 2 - 1 - 1 - 1)^2 > 0,$$

$$|t_{1112} - t_{1222}| = 2 < \sqrt{6t_{1122} + 2} = \sqrt{8}.$$

So $\mathcal{T}$ is positive semidefinite if and only if $t_{1122} = 1$.

(ii) It follows from Lemma 2.1 (I) that $\mathcal{T}$ is positive definite if and only if

$$I^3 - 27J^2 = 0,\ \ t_{1112} = t_{1222},\ \ 2t_{1112}^2 + 1 = 3t_{1122} < 3;$$
$$I^3 - 27J^2 > 0,\ \ |t_{1112} - t_{1222}| \le \sqrt{6t_{1122} + 2} \text{ and } -1 < 3t_{1122} \le 3.$$

Since $3 = 2t_{1112}^2 + 1 = 3t_{1122} < 3$ can't hold, then $\mathcal{T}$ is positive definite if and only if $t_{1122} = 1$ and $t_{1112}t_{1222} = -1$. This completes the proof.

3. Positive definiteness of 4th order 3-dimensional symmetric tensors

Theorem 3.1. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with its entries,

$$|t_{iiij}| = |t_{iijk}| = t_{iiii} = 1 \text{ and } t_{ijjj}t_{iiij} = -1 \text{ for all } i, j, k \in \{1, 2, 3\},\ i \neq j,\ i \neq k,\ j \neq k.$$

(i) If $t_{iijj} = \frac{11}{6}$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$, then $\mathcal{T}$ is positive semidefinite if and only if

(III) $\quad t_{1222} = t_{2333} = t_{1113}, t_{1112} = t_{1333} = t_{2223}, t_{1123} = t_{1223} = t_{1233} = -1.$

(ii) If $t_{iijj} = 2$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$, then $\mathcal{T}$ is positive definite if and only if the above condition (**III**) holds.

(iii) If $t_{iijj} = 2.5$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$, then $\mathcal{T}$ is positive definite if and only if

$$\textbf{(IV)} \qquad \begin{cases} t_{1123} = t_{1223} = t_{1233} = 1; \text{ or} \\ t_{1123} = t_{1223} = t_{1233} = -1, t_{1222} = t_{2333} = t_{1113}, t_{1112} = t_{1333} = t_{2223}; \text{ or} \\ \text{two of } \{t_{1123}, t_{1223}, t_{1233}\} \text{ are } -1. \end{cases}$$

(iv) If $t_{iijj} \geq \frac{8}{3}$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$, then $\mathcal{T}$ is positive definite.

Proof. $\mathcal{T}x^4$ may be rewritten as follows,

$$\begin{aligned}
\mathcal{T}x^4 =&(x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3)\\
&+ 6(t_{1122} - 1)x_1^2x_2^2 + 6(t_{1133} - 1)x_1^2x_3^2 + 6(t_{2233} - 1)x_2^2x_3^2\\
&+ 12(t_{1123} - 1)x_1^2x_2x_3 + 12(t_{1223} - 1)x_1x_2^2x_3 + 12(t_{1233} - 1)x_1x_2x_3^2\\
=&(x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2)\\
&+ 6(t_{1122} - 1)x_1^2x_2^2 + 6(t_{1133} - 1)x_1^2x_3^2 + 6(t_{2233} - 1)x_2^2x_3^2\\
&+ 12(t_{1123} + 1)x_1^2x_2x_3 + 12(t_{1223} + 1)x_1x_2^2x_3 + 12(t_{1233} - 1)x_1x_2x_3^2\\
=&(x_1 - x_2 + x_3)^4 + 8(x_1x_2^3 + x_2x_3^3 - x_1^3x_3)\\
&+ 6(t_{1122} - 1)x_1^2x_2^2 + 6(t_{1133} - 1)x_1^2x_3^2 + 6(t_{2233} - 1)x_2^2x_3^2\\
&+ 12(t_{1123} + 1)x_1^2x_2x_3 + 12(t_{1223} - 1)x_1x_2^2x_3 + 12(t_{1233} + 1)x_1x_2x_3^2\\
=&(x_2 + x_3 - x_1)^4 + 8(x_1^3x_3 + x_1x_2^3 - x_2^3x_3)\\
&+ 6(t_{1122} - 1)x_1^2x_2^2 + 6(t_{1133} - 1)x_1^2x_3^2 + 6(t_{2233} - 1)x_2^2x_3^2\\
&+ 12(t_{1123} - 1)x_1^2x_2x_3 + 12(t_{1223} + 1)x_1x_2^2x_3 + 12(t_{1233} + 1)x_1x_2x_3^2.
\end{aligned}$$

(i) Necessity. Suppose the conditions (**III**) can't hold when $\mathcal{T}$ is positive semidefinite, then there may be four cases.

Case 1. There is two -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$. We might as well take $t_{1123} = t_{1223} = -1$ and $t_{1233} = 1$. Without loss the generality, let $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. Take $x = (\frac{1}{5}, -\frac{1}{5}, 1)^\top$. Then we have

$$\begin{aligned}
\mathcal{T}x^4 =&(x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1x_2^3) + 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)\\
=&1 + 8\left(\frac{1}{5^3} - \frac{1}{5} + \frac{1}{5^4}\right) + 5\left(\frac{1}{5^4} + \frac{1}{5^2} + \frac{1}{5^2}\right) = -\frac{72}{625} < 0;
\end{aligned}$$

Case 2. There is only one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$. We might as well take $t_{1123} = t_{1233} = 1$ and $t_{1223} = -1$ and $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. For $x = (\frac{1}{2}, -\frac{1}{2}, 1)^\top$, we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 + x_3)^4 - 8(x_1x_2^3 + x_1x_3^3 + x_2^3x_3) + 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2^2x_3\\ =&1 - 8\left(-\frac{1}{2^4} + \frac{1}{2} - \frac{1}{2^3}\right) + 5\left(\frac{1}{2^4} + \frac{1}{2^2} + \frac{1}{2^2}\right) - 3 = -\frac{27}{16} < 0.\end{aligned}$$

Case 3. $t_{1123} = t_{1223} = t_{1233} = 1$ and $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. Take $x = (\frac{1}{5}, -\frac{1}{5}, 1)^\top$. Then we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 + x_3)^4 - 8(x_1x_2^3 + x_1x_3^3 + x_2^3x_3) + 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)\\ =&1 - 8\left(-\frac{1}{5^4} + \frac{1}{5} - \frac{1}{5^3}\right) + 5\left(\frac{1}{5^4} + \frac{1}{5^2} + \frac{1}{5^2}\right) = -\frac{72}{625} < 0.\end{aligned}$$

The above three cases imply that the equality, $t_{1123} = t_{1223} = t_{1233} = -1$, is necessary.

Case 4. $t_{1123} = t_{1223} = t_{1233} = -1$, but $t_{1222} = t_{2333} = t_{1113}$ and $t_{1112} = t_{1333} = t_{2223}$ can't hold. Without loss the generality, let $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. Take $x = (-1, -3, -1)^\top$. Then we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1x_2^3) + 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2\\ =&(-1 - 3 + 1)^4 - 8(1 + 3 - 3^3) + 5(9 + 1 + 9) - 24 \times 3 = -80 < 0.\end{aligned}$$

This is a contradiction to the positive semidefiniteness of $\mathcal{T}$, and hence, the conditions (**III**) are necessary.

Sufficiency. $t_{1123} = t_{1223} = t_{1233} = -1$ and $t_{1222} = t_{2333} = t_{1113}$ and $t_{1112} = t_{1333} = t_{2223}$. Without loss the generality, let $t_{1222} = t_{2333} = t_{1113} = 1$ and $t_{1112} = t_{1333} = t_{2223} = -1$. Rewriting $\mathcal{T}x^4$ as follow,

$$\mathcal{T}x^4 = (x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) + 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2.$$

Solve the constrained optimization problem:

$$\begin{aligned}&\min\ \mathcal{T}x^4\\ &\text{s. t. } x_1^2 + x_2^2 + x_3^2 = 1.\end{aligned}$$

Then the minimum value is 0 at a point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ or $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, and hence, $\mathcal{T}x^4 \geq 0$. That is, $\mathcal{T}$ is positive semidefinite.

(ii) Necessity. Suppose the conditions (**III**) can't hold when $\mathcal{T}$ is positive definite, then there may be four cases.

Case 1. There is two -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$. We might as well take $t_{1123} = t_{1223} = -1$ and $t_{1233} = 1$ and $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. Take $x = (\frac{1}{5}, -\frac{1}{5}, 1)^\top$. Then we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1x_2^3) + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)\\ =&1 + 8\left(\frac{1}{5^3} - \frac{1}{5} + \frac{1}{5^4}\right) + 6\left(\frac{1}{5^4} + \frac{1}{5^2} + \frac{1}{5^2}\right) = -\frac{21}{625} < 0;\end{aligned}$$

Case 2. There is only one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$. We might as well take $t_{1123} = t_{1233} = 1$ and $t_{1223} = -1$ and $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. For $x = (\frac{1}{2}, -\frac{1}{2}, 1)^\top$, we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 + x_3)^4 - 8(x_1x_2^3 + x_1x_3^3 + x_2^3x_3) + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2^2x_3\\ =&1 - 8\left(-\frac{1}{2^4} + \frac{1}{2} - \frac{1}{2^3}\right) + 6\left(\frac{1}{2^4} + \frac{1}{2^2} + \frac{1}{2^2}\right) - 3 = -\frac{9}{8} < 0.\end{aligned}$$

Case 3. $t_{1123} = t_{1223} = t_{1233} = 1$ and $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. Take $x = (\frac{1}{5}, -\frac{1}{5}, 1)^\top$. Then we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 + x_3)^4 - 8(x_1x_2^3 + x_1x_3^3 + x_2^3x_3) + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)\\ =&1 - 8\left(-\frac{1}{5^4} + \frac{1}{5} - \frac{1}{5^3}\right) + 6\left(\frac{1}{5^4} + \frac{1}{5^2} + \frac{1}{5^2}\right) = -\frac{21}{625} < 0.\end{aligned}$$

The above three cases imply that the equality, $t_{1123} = t_{1223} = t_{1233} = -1$, is necessary.

Case 4. $t_{1123} = t_{1223} = t_{1233} = -1$, but $t_{1222} = t_{2333} = t_{1113}$ and $t_{1112} = t_{1333} = t_{2223}$ can't hold. Without loss the generality, let $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. Take $x = (-1, -3, -1)^\top$. Then we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1x_2^3) + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2\\ =&(-1 - 3 + 1)^4 - 8(1 + 3 - 3^3) + 6(9 + 1 + 9) - 24 \times 3 = -61 < 0.\end{aligned}$$

This is a contradiction to the positive definiteness of $\mathcal{T}$, and hence, the conditions (**III**) are necessary.

Sufficiency. $t_{1123} = t_{1223} = t_{1233} = -1$, $t_{1222} = t_{2333} = t_{1113}$ and $t_{1112} = t_{1333} = t_{2223}$. Without loss the generality, let $t_{1222} = t_{2333} = t_{1113} = 1$ and $t_{1112} = t_{1333} = t_{2223} = -1$. Rewriting $\mathcal{T}x^4$ as follow,

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2\\ \geq&(x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) + 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2.\end{aligned}$$

By (i), $\mathcal{T}x^4 \geq 0$ for all $x \in \mathbb{R}^3$. It is not difficult to verify that the solutions of the equation $\mathcal{T}x^4 = 0$ is only original point $O(0, 0, 0)$. So, $\mathcal{T}x^4 > 0$ for all $x \neq 0$. That is, $\mathcal{T}$ is positive definite.

(iii) Assume $t_{iijj} = 2.5$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$. If the conditions **(IV)** can't hold when $\mathcal{T}$ is positive definite, then there are only two cases.

Case 1. one of $\{t_{1123}, t_{1223}, t_{1233}\}$ is only -1. We might be $t_{1123} = t_{1233} = 1$ and $t_{1223} = -1$ and $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. Then for $x = (\frac{1}{4}, -\frac{1}{4}, 1)^\top$, we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 + x_3)^4 - 8(x_1x_2^3 + x_1x_3^3 + x_2^3x_3) + 9(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2^2x_3\\ =&1 - 8\left(-\frac{1}{4^4} + \frac{1}{4} - \frac{1}{4^3}\right) + 9\left(\frac{1}{4^4} + \frac{1}{4^2} + \frac{1}{4^2}\right) - 24 \times \frac{1}{4^3} = -\frac{15}{256} < 0.\end{aligned}$$

Case 2. $t_{1123} = t_{1223} = t_{1233} = -1$, but $t_{1222} = t_{2333} = t_{1113}$ and $t_{1112} = t_{1333} = t_{2223}$ can't hold. Without loss the generality, let $t_{1112} = t_{2333} = t_{1113} = 1$ and $t_{1222} = t_{1333} = t_{2223} = -1$. Take $x = (-1, -3, -1)^\top$. Then we have

$$\begin{aligned}\mathcal{T}x^4 =&(x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1x_2^3) + 9(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2\\ =&(-1 - 3 + 1)^4 - 8(1 + 3 - 3^3) + 9(9 + 1 + 9) - 24 \times 3 = -4 < 0.\end{aligned}$$

This obtains a contracdiction, and so, the conditions **(IV)** is necessary.

Now we show the sufficiency. The second condition easily established by the proof of (ii), we only show the conclusion holds under the conditions: $t_{1123} = t_{1223} = t_{1233} = 1$ and two of $\{t_{1123}, t_{1223}, t_{1233}\}$ are -1. Let $t_{1222} = t_{2223} = t_{1113} = 1$ and $t_{1112} = t_{1333} = t_{2333} = -1$.

Condition: $t_{1123} = t_{1223} = t_{1233} = 1$. Rewriting $\mathcal{T}x^4$ as follow,

$$\mathcal{T}x^4 = (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2x_3^3) + 9(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2).$$

It is easy to verify the global minimum problem, $\min \mathcal{T}x^4$ has unique minimum 0 at the origin coordinates $O(0, 0, 0)$. So, $\mathcal{T}$ is positive definite.

Condition: two of $\{t_{1123}, t_{1223}, t_{1233}\}$ are -1. Without loss the generality, take $t_{1123} = t_{1223} = -1$ and $t_{1233} = 1$. Then we have

$$\mathcal{T}x^4 = (x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2^3x_3 - x_1^3x_2) + 9(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2).$$

Solve the global minimum problem, $\min \mathcal{T}x^4$, to yield its minimum value 0 at the origin coordinates $O(0, 0, 0)$. So, $\mathcal{T}$ is positive definite.

(iv) Assume $t_{iijj} \geq \dfrac{8}{3}$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$. Obvously, the condition $|t_{iijk}| = 1$ for all $i, j, k \in \{1, 2, 3\}, i \neq j, i \neq k, k \neq j$, is equivalent to

$$\textbf{(V)} \qquad \begin{cases} t_{1123} = t_{1223} = t_{1233} = 1; \text{ or} \\ t_{1123} = t_{1223} = t_{1233} = -1; \text{ or} \\ \text{two of } \{t_{1123}, t_{1223}, t_{1233}\} \text{ are } -1; \text{ or} \\ \text{one of } \{t_{1123}, t_{1223}, t_{1233}\} \text{ are } -1. \end{cases}$$

We need only show the conclusion holds under the conditions: one of $\{t_{1123}, t_{1223}, t_{1233}\}$ is -1 or $t_{1123} = t_{1223} = t_{1233} = -1$. Other two cases directly follow from (iii). Without loss the generality, let $t_{1222} = t_{2223} = t_{1113} = 1$ and $t_{1112} = t_{1333} = t_{2333} = -1$.

Assume one of $\{t_{1123}, t_{1223}, t_{1233}\}$ is only -1. We might take $t_{1123} = t_{1223} = 1$ and $t_{1233} = -1$. Then we have

$$\mathcal{T}x^4 \geq (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2x_3^3) + 10(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2.$$

Let

$$g(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2x_3^3) + 10(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2.$$

Solve the global minimum problem,

$$\min \left\{ g(x_1, x_2, x_3); x = (x_1, x_2, x_3)^\top \in \mathbb{R}^n \right\}$$

to yield its minimum value 0 at the origin coordinates $O(0, 0, 0)$. So, $\mathcal{T}x^4 \geq g(x_1, x_2, x_3) > 0$ for all $x \in \mathbb{R}^3 \setminus \{0\}$. Thus $\mathcal{T}$ is positive definite.

Assume $t_{1123} = t_{1223} = t_{1233} = -1$. Then we have

$$\mathcal{T}x^4 \geq (x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) + 10(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2.$$

Let

$$f(x_1, x_2, x_3) = (x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) + 10(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2.$$

Solve the global minimum problem,

$$\min\left\{f(x_1,x_2,x_3); x=(x_1,x_2,x_3)^\top\in\mathbb{R}^n\right\}$$

to yield its minimum value 0 at the origin coordinates $O(0,0,0)$. So, $\mathcal{T}x^4 \geq f(x_1,x_2,x_3) > 0$ for all $x\in\mathbb{R}^3\setminus\{0\}$. That is, $\mathcal{T}$ is positive definite. This completes the proof.

By applying Theorems 3.1 (i) and (ii), the following inequalities are established easily for ternary quartic homogeneous polynomials.

Corollary 3.2. *If $(x_1,x_2,x_3)\neq(0,0,0)$, then*

(i) $(x_1+x_2-x_3)^4+5(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)\geq 8(x_1^3x_2-x_1^3x_3-x_2x_3^3)+24x_1x_2x_3^2$, *with equality if and only if $x_1=x_2=x_3$;*

(ii) $(x_1+x_2-x_3)^4+6(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)> 8(x_1^3x_2-x_1^3x_3-x_2x_3^3)+24x_1x_2x_3^2$.

Furthermore, these (strict) inequalities still hold if $x_1^3x_2$ and $x_1x_2^3$, $x_1^3x_3$ and $x_1x_3^3$, $x_2x_3^3$ and $x_2^3x_3$ are simultaneously exchangeable.

By applying Theorems 3.1 (iii) and (iv), the following strict inequalities are established easily for ternary quartic homogeneous polynomials.

Corollary 3.3. *If $(x_1,x_2,x_3)\neq(0,0,0)$, then*

(i) $(x_1+x_2+x_3)^4+9(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)> 8(x_1x_3^3+x_1^3x_2+x_2^3x_3)$;

(ii) $(x_1+x_2+x_3)^4+10(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)> 8(x_1x_3^3+x_1^3x_2+x_2^3x_3)+24x_1x_2x_3^2$;

(iii) $(x_1+x_2-x_3)^4+10(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)> 8(x_1^3x_2-x_1^3x_3-x_2x_3^3)+24x_1x_2x_3^2$;

(iv) $(x_1+x_2-x_3)^4+9(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)> 8(x_1^3x_2-x_1^3x_3-x_2x_3^3)$.

Furthermore, these strict inequalities still hold if $x_1^3x_2$ and $x_1x_2^3$ are exchangeable, or $x_1^3x_3$ and $x_1x_3^3$ are exchangeable, or $x_2x_3^3$ and $x_2^3x_3$ are exchangeable.

4. Conclusions

For a 4th order 3-dimensional symmetric tensor with its entries 1 or -1, the analytic necessary and sufficient conditions are established for its positive definiteness. Several (strict) inequalities of ternary quartic homogeneous polynomial are built by means of these analytic conditions.

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